Clocking the End of Cosmic Inflation

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Abstract. Making observable predictions for cosmic inflation requires determining when the wavenumbers of astrophysical interest today exited the Hubble radius during the inflationary epoch. These instants are commonly evaluated using the slow-roll approximation and measured in e-folds $\Delta N = N - N_{\text{end}}$, in reference to the e-fold N_{end} at which inflation ended. Slow roll being necessarily violated towards the end of inflation, both the approximated trajectory and N_{end} are determined at, typically, one or two e-folds precision. Up to now, such an uncertainty has been innocuous, but this will no longer be the case with the forthcoming cosmological measurements. In this work, we introduce a new and simple analytical method, on top of the usual slow-roll approximation, that reduces uncertainties on ΔN to less than a tenth of an e-fold.

Keywords: Cosmic Inflation, Corrections to slow-roll

Contents

1 Introduction

Cosmic Inflation is a phase of accelerated expansion of the primordial universe which addresses various puzzles of the Big-Bang model as, for instance, the so-called horizon problem and the smallness of the spatial curvature today [1–12]. Inflation also provides a convincing and simple physical explanation for the origin of cosmic structures: they are seeded by vacuum quantum fluctuations of both the metric and a yet unknown scalar degree of freedom [13].

In its simplest incarnation, both the accelerated expansion and the quantum fluctuations are the outcome of a self-gravitating scalar field ϕ, named the inflaton, slowly rolling down its potential energy $V(\phi)$. This class of scenarios is a populated landscape counting hundreds of models, all of them making definite predictions which can be confronted by cosmological observations [14–16]. As of today, 40% of the proposed scenarios in this class have been ruled-out by Cosmic Microwave Background (CMB) and Large Scale Structure measurements [17–19]. Still, 60% of the remaining models are compatible with the data. With the deployment of ground-based CMB-S4 polarization telescopes [20–22], the soon-to-be released Euclid satellite data [23, 24], unprecedented galaxy surveys [25], and the search for B-mode polarization from space by the LiteBIRD satellite [26], one should reasonably expect many of the remaining models to be disambiguated and tested. This necessitates, however, that theoretical predictions are made at the required accuracy [27, 28].

In a brute-force manner, one can simply solve the field and gravitational evolution numerically. However, the underlying gravity theory is General Relativity, and, as of today, only parts of the inflationary evolution can be solved without approximation [29–36]. Considering the regime in which the quantum fluctuations do not dominate the dynamics [37–42], one can alternatively solve for linear and non-linear perturbations numerically, around a homogeneous background, without any other approximations [43–51]. These methods are accurate as long as the gravitational effects remain small, but they are computationally too much demanding when dealing with hundreds of different models [52, 53].

There exists, however, a model-free and perturbative treatment for the single-field scenarios in which the assumption of "slow roll" can be made. This approach, initiated in Ref. [1] for the tensor modes, has been extended to scalar perturbations in Refs. [54, 55] and generalized to higher orders in [56–63]. It has also found applications out of the original context and can be extended to other classes of inflationary models [64–85]. In modern terminology, the slow-roll approximation introduces the Hubble-flow functions defined by [59]

$$\epsilon_{i+1}(N) \equiv \frac{\mathrm{d}\ln|\epsilon_i|}{\mathrm{d}N}\,, \quad \epsilon_0(N) = \frac{M_{\mathrm{Pl}}}{H}\,, \tag{1.1}$$

where $H(N)$ denotes the Hubble parameter during inflation and $N = \ln a$ is the number of e-folds, a being the Friedmann-Lemaître-Robertson-Walker (FLRW) scale factor. For a quasi-de Sitter accelerated expansion, $H(N)$ is nearly constant and all the Hubble-flow functions are expected to be small. It is therefore possible to solve the linearized Einstein's equations for both the tensor and scalar perturbations by performing a consistent expansion in terms of these ϵ_i functions. Analytical solutions have currently been derived up to third order [63] and they allow us to calculate the primordial power spectra of the comoving curvature perturbation ζ and of the primordial gravitational waves $h_{\mu\nu}$. For instance, keeping only the first order terms, one gets, for the power spectrum of the curvature perturbations,

$$\mathcal{P}_\zeta(k) = \frac{H_*^2}{8\pi^2 M_{\mathrm{Pl}}^2 \epsilon_{1*}}\left[1 - 2(C+1)\epsilon_{1*} - C\epsilon_{2*} - (2\epsilon_{1*}+\epsilon_{2*})\ln\left(\frac{k}{k_*}\right) + \dots\right], \tag{1.2}$$

where the constant $C \equiv \gamma_{\mathrm{E}} + \ln(2) - 2 \simeq -0.7296$ and k_* is a wavenumber around which the expansion is made (an observer choice). For the wavenumbers probed by the Cosmic Microwave Background anisotropies, one usually takes $k_*/a_0 = 0.05\,\mathrm{Mpc}^{-1}$ to be in the middle of the observable range of modes. All the other "starred" quantities in Eq. (1.2) refer to the Hubble-flow functions evaluated at a given e-fold number N_*, i.e., $\epsilon_{i*} = \epsilon_i(N_*)$ and $H_* = H(N_*)$. This e-fold number is the time at which the physical pivot wavenumber k_*/a exited the Hubble radius during inflation, namely the solution of[1]

$$k_* \simeq a(N_*)H(N_*). \tag{1.3}$$

As a result, even if the accuracy at which Eq. (1.2) is derived is under control, another source of uncertainties in making observable predictions comes from our ability to determine a precise value for N_*, and, as we will see, for $\Delta N_* \equiv N_* - N_{\mathrm{end}}$. This is the main focus of this paper.

Before entering into details, let us further express Eq. (1.3) in terms of observable quantities. The physical pivot wavenumber is measured today, for a scale factor given by a_0,

[1]For practical reasons, N_* is usually defined in terms of the conformal time η by $k_*\eta(N_*) = -1$, which coincides with Hubble radius crossing $k_* = a(N_*)H(N_*)$ at leading order in the Hubble-flow functions. For the present discussion, these differences will not play a role, but they are important when considering higher order terms [63].

in terms of which Eq. (1.3) reads

$$\frac{k_*}{a_0} = (1 + z_{\rm end})^{-1} \frac{a(N_*)}{a(N_{\rm end})} H(N_*). \tag{1.4}$$

We have made explicit $z_{\rm end} = a_0/a_{\rm end} - 1$, the redshift at which inflation ended. It depends on the universe history *after* inflation and, in particular, it is sensitive to the so-called reheating era. Following Refs. [52, 86], one can conveniently absorb all the kinematic effects associated with this era into the *reheating parameter* $R_{\rm rad}$ defined by

$$R_{\rm rad} \equiv \frac{a_{\rm end}}{a_{\rm reh}} \left(\frac{\rho_{\rm end}}{\rho_{\rm reh}}\right)^{1/4}. \tag{1.5}$$

Here $\rho_{\rm end}$ and $\rho_{\rm reh}$ stand for the energy density of the universe at the end of inflation and at beginning of the radiation era (the end reheating), respectively. In terms of $R_{\rm rad}$, one has

$$1 + z_{\rm end} = \frac{1}{R_{\rm rad}} \left(\frac{\rho_{\rm end}}{3\mathcal{Q}_{\rm reh}\Omega_{\rm rad} H_0^2}\right)^{1/4}, \tag{1.6}$$

where $\mathcal{Q}_{\rm reh} \equiv q_0^{4/3} g_{\rm reh}/(q_{\rm reh}^{4/3} g_0)$ is a measure of the change of number of entropic (q) and energetic (g) relativistic degrees of freedom between the beginning of the radiation era and today [87]. For instance, one has $\mathcal{Q}_{\rm reh} \simeq 0.39$ for the Standard Model [88]. One can further expand Eqs. (1.4) to (1.6) for single-field inflationary models by making use of the Friedmann-Lemaître equations for a self gravitating scalar field ϕ. As shown in Section 2, they allow us to express the Hubble parameter during inflation as

$$H^2 = \frac{\rho}{3M_{\rm Pl}^2} = \frac{1}{M_{\rm Pl}^2} \frac{V(\phi)}{3-\epsilon_1}. \tag{1.7}$$

At the end of inflation, one therefore has

$$\rho_{\rm end} = \frac{3V_{\rm end}}{3-\epsilon_{1\rm end}} = \frac{V_{\rm end}}{V_*}\frac{3V_*}{3-\epsilon_{1\rm end}} = 3M_{\rm Pl}^2 H_*^2 \frac{V_{\rm end}}{V_*}\frac{3-\epsilon_{1*}}{3-\epsilon_{1\rm end}}. \tag{1.8}$$

Plugging this expression into Eq. (1.6), taking the logarithm of Eq. (1.4), one finally gets

$$\Delta N_* \equiv N_* - N_{\rm end} = -\ln R_{\rm rad} + N_0 + \frac{1}{4}\ln\left[\frac{9}{\epsilon_{1*}\left(3-\epsilon_{1\rm end}\right)}\frac{V_{\rm end}}{V_*}\right] - \frac{1}{4}\ln\left(8\pi^2 P_0\right), \tag{1.9}$$

where N_0 is defined by

$$N_0 \equiv \ln\left[\frac{k_*/a_0}{\left(3\mathcal{Q}_{\rm reh}\Omega_{\rm rad}H_0^2 M_{\rm Pl}^2\right)^{1/4}}\right] \simeq -61.5, \tag{1.10}$$

the absolute value of which giving the typical number of e-folds of decelerated expansion after inflation. In all practical situations, one has $\epsilon_{i*} \ll 1$ and, for consistency, we have kept only the leading order terms in ϵ_{i*} while deriving Eq. (1.9). Moreover, in the last term, we have made explicit the quantity $P_0 = H_*^2/(8\pi^2 M_{\rm Pl}^2 \epsilon_{1*})$, which is a very well measured observable as $P_0 \simeq \mathcal{P}_\zeta(k_*) = 2.097 \times 10^{-9}$ [17].

Let us now explain how to determine the value of N_* under the hypothesis that an inflationary model, given by its potential $V(\phi)$, is specified. Any possible reheating history is associated with definite values for $\ln R_{\rm rad}$ (and N_0). For instance, a radiation-like, or an instantaneous reheating, are both associated with $\ln R_{\rm rad} = 0$. In this situation, Eq. (1.9) ends up being a simple algebraic equation for N_* provided one has an explicit expression for $V_* = V[\phi(N_*)]$, $V_{\rm end} = V[\phi(N_{\rm end})]$ and $N_{\rm end}$. In other words, one must determine the field trajectory $\phi(N)$ to solve Eq. (1.9) at given reheating history. Usually, this cannot be made exactly and one has to resort to an exact numerical integration, or, to some slow roll approximation to evaluate the field trajectory $\phi(N)$. Notice that, it is also possible to interpret Eq. (1.9) as an algebraic equation on $\phi_* = \phi(N_*)$, but this still requires determining the field trajectory in order to evaluate $N_* = N(\phi_*)$.

The fastest and most practical method used to determine the field trajectory is the slow-roll approximation. As we show in Section 2.2, it induces $\mathcal{O}(1)$ errors, which have been, up to now, not a concern. Indeed, most of the theoretical unknowns in Eq. (1.9) are actually associated with the reheating, namely the values of $\ln R_{\rm rad}$ ($\mathcal{Q}_{\rm reh}$ does not have significant effects provided the number of relativistic degrees of freedom does not take exponentially large values [89]). The actual value of $\rho_{\rm reh}$ is unknown by orders of magnitude and, in principle, it is allowed to vary from a lower bound as small as Big-Bang Nucleosynthesis $\rho_{\rm nuc}^{1/4} = \mathcal{O}(\mathrm{MeV})$ to $\rho_{\rm end}^{1/4}$ which can be as large as $10^{15}\,\mathrm{GeV}$. Under very reasonable assumptions, one can show that $\ln R_{\rm rad} \in [-46, 15]$ (see Ref. [52, 90]).

This justifies why questioning the accuracy at which $N(\phi)$ is evaluated was not a concern. However, as shown in Ref. [19], the current cosmological data are now constraining the reheating era and models having exactly the same accelerated inflationary phase but differing only by their reheating histories, i.e., predicting different values of $\ln R_{\rm rad}$, can now be disambiguated. From another point of view, even for inflationary scenarios not specifying the reheating, the current data allow us to determine the favoured values of $\ln R_{\rm rad}$. Any uncertainty in the determination of ΔN_* will then bias the constraints on $\ln R_{\rm rad}$. As such, it is becoming relevant to improve the accuracy at which the function $N(\phi)$ can be determined.

The paper is organized as follows. In Section 2, we recap how to obtain the field trajectory within a FLRW metric and detail the slow-roll method commonly used to approximate the solution. In particular, we use a numerical integration to discuss the amplitude and the origin of the uncertainties made by using the slow-roll approximated trajectory instead of the exact one. In Section 3, we present new analytical results and an exact expansion of the trajectory which allow us to propose a "velocity correction" to the traditional slow-roll. We show that such a correction reduces the uncertainties by an order of magnitude. Section 4 is dedicated to the problem of determining the field value $\phi_{\rm end}$ at which inflation ends, which is another (small) source of errors on the determination of $\rho_{\rm end}$. We present various analytical approaches to address this issue and test them within various inflationary scenarios. Here as well, we show that our method reduces the uncertainties on $\phi_{\rm end}$ by an order of magnitude. Our conclusion are presented in Section 5.

2 Basics on the field trajectory

2.1 Equations of motion

In the following, we assume a minimally coupled single scalar field ϕ within a FLRW metric. The Friedmann-Lemaître and Klein-Gordon equations read

$$H^2 = \frac{1}{3M_{\mathrm{Pl}}^2}\left[\frac{1}{2}\dot{\phi}^2 + V(\phi)\right], \tag{2.1}$$

$$H^2 + \dot{H}^2 = -\frac{1}{6M_{\mathrm{Pl}}^2}\left[2\dot{\phi}^2 - 2V(\phi)\right], \tag{2.2}$$

$$\ddot{\phi} + 3H\dot{\phi} + \frac{\mathrm{d}V}{\mathrm{d}\phi} = 0, \tag{2.3}$$

where a dot denotes differentiation with respect to the cosmic time t and $H \equiv \dot{a}/a$. In terms of the number of e-fold $N \equiv \ln a$, these equations can be decoupled. From Eqs. (1.1), (2.1) and (2.2), one has

$$\epsilon_1 = -\frac{\dot{H}}{H^2} = \frac{1}{2M_{\mathrm{Pl}}^2}\left(\frac{\mathrm{d}\phi}{\mathrm{d}N}\right)^2, \tag{2.4}$$

and the first Hubble-flow function ϵ_1 measures the kinetic energy of the field when time is counted in e-fold. In order to simplify the notations, let us introduce the "field velocity" in e-fold as

$$\Gamma \equiv \frac{1}{M_{\mathrm{Pl}}}\frac{\mathrm{d}\phi}{\mathrm{d}N} = \frac{1}{M_{\mathrm{Pl}}H}\dot{\phi}, \tag{2.5}$$

such that $\epsilon_1 = \Gamma^2/2$. Expressing Eqs. (2.1) and (2.2) in e-fold time, one obtains Eq. (1.7) for the Hubble parameter, which can be finally plugged into Eq. (2.3) to obtain a decoupled equation of motion for $\phi(N)$

$$\frac{1}{3-\epsilon_1}\frac{\mathrm{d}^2\phi}{\mathrm{d}N^2} + \frac{\mathrm{d}\phi}{\mathrm{d}N} = -M_{\mathrm{Pl}}^2\frac{\mathrm{d}\ln V}{\mathrm{d}\phi}. \tag{2.6}$$

From now on, we will be working in Planck units with $M_{\mathrm{Pl}} = 1$ such that, making use of Eq. (2.4), the previous equation simplifies to

$$\frac{2}{6-\Gamma^2}\frac{\mathrm{d}\Gamma}{\mathrm{d}N} + \Gamma = -\frac{\mathrm{d}\ln V}{\mathrm{d}\phi}. \tag{2.7}$$

As discussed in Ref. [18], this equation is similar to the one of a relativistic particle in presence of friction and accelerated by an external force created by the potential $W(\phi) = \ln[V(\phi)]$, the value $\sqrt{6}$ giving the maximal possible speed for the field ϕ. Indeed, positivity of Eq. (1.7) enforces that all field trajectories must satisfy $\epsilon_1 < 3$, i.e., $|\Gamma| < \sqrt{6}$.

2.2 Slow-roll trajectory

There is no known analytical solution of Eq. (2.7) for an unspecified potential $V(\phi)$, although various approximated solutions, in different regimes, have been derived [18] (see, however, Appendix A). In the slow-roll regime we are interested in, one can remark that the field acceleration can also be expressed in terms of ϵ_2. From Eqs. (1.1) and (2.5), one has

$$\frac{\mathrm{d}\Gamma}{\mathrm{d}N} = \frac{1}{2}\epsilon_2\Gamma, \tag{2.8}$$

in terms of which Eq. (2.7) reads

$$\left(1+\frac{\epsilon_2}{6-2\epsilon_1}\right)\Gamma = -\frac{\mathrm{d}\ln V}{\mathrm{d}\phi}\,. \tag{2.9}$$

Assuming a slowly rolling field evolution implies that all the ϵ_i are small and, at leading order, Eq. (2.9) can be approximated by

$$\Gamma \simeq \Gamma_{\mathrm{sr}} \equiv -\frac{\mathrm{d}\ln V}{\mathrm{d}\phi}, \tag{2.10}$$

which has the solution

$$N_{\mathrm{sr}}(\phi) = -\int^{\phi} \frac{V(\psi)}{V'(\psi)}\,\mathrm{d}\psi\,. \tag{2.11}$$

Here the prime stands for the derivative with respect to field value. Another way to interpret this slow-roll trajectory is to remark that the acceleration term of Eq. (2.7) is ignored, which means that we are only considering the friction dominated regime. In fact, were the force term on the right-hand-side be constant, Eq. (2.10) would give the exact terminal velocity of Eq. (2.7), and, Eq. (2.11) would be the exact attractor solution. In the general case, however, there is a small drift sourced by the non-constancy of the force term and this induces differences between $N_{\mathrm{sr}}(\phi)$ and the exact attractor solution $N(\phi)$ of Eq. (2.7).

Let us remark that Eq. (2.11) is defined up to a constant term. However, as explained in Section 1, the quantity of interest for inflation is ΔN_* and only the functional $\Delta N(\phi) = N(\phi) - N_{\mathrm{end}}$, in which a possible constant term cancels, is observable. As such, in addition to $N_{\mathrm{sr}}(\phi)$, one should also estimate N_{end} accurately.

2.3 Characterizing the end of inflation

By definition, inflation stands for an accelerated expansion of the spacetime, i.e., $\ddot{a} > 0$. From Eq. (2.4), using $H = \dot{a}/a$, one has $\epsilon_1 = 1 - \ddot{a}/(aH^2)$ and the condition for acceleration translates into $\epsilon_1 < 1$. In the vanilla single-field inflationary models, the accelerated expansion ends by itself with a so-called "graceful exit": the potential becomes steeper, and the field accelerates up to the point at which

$$\epsilon_{1\mathrm{end}} \equiv \epsilon_1(N_{\mathrm{end}}) = 1. \tag{2.12}$$

Translated into velocities, one therefore has $\Gamma_{\mathrm{end}} \equiv \Gamma(N_{\mathrm{end}}) = \pm\sqrt{2}$, the sign being related to the direction in which inflation proceeds. Indeed, depending on the shape of the potential, either the field increases during inflation and $\Gamma > 0$, or it decreases and $\Gamma < 0$. It is also possible that inflation ends by another mechanism than a graceful exit, as for instance by a tachyonic instability triggered by an extra field, as in the prototypical hybrid inflation model [91]. In that situation, N_{end} is no longer set by the condition $\epsilon_1(N_{\mathrm{end}}) = 1$. Instead, it may be viewed as an additional model parameter. The determination of N_{end} in these situations has, therefore, nothing to do with the inflationary dynamics, and we will not consider these cases. Let us however stress that a tachyonic instability is relevant only if it affects the inflaton while $\epsilon_1 < 1$, otherwise it would rather be interpreted as an event belonging to the reheating era.

Solving Eq. (2.12) for N_{end} is problematic in various aspects. It is a condition on the first Hubble-flow function, or equivalently, on the field velocity Γ, whose e-fold dependency would require to solve Eq. (2.7) exactly. Without knowing the exact solution, the best one can do is to use the approximation of Eq. (2.10) and solve $\Gamma_{\rm sr}(N_{\rm end}^{\rm sr}) = \pm\sqrt{2}$ instead of $\Gamma(N_{\rm end}) = \pm\sqrt{2}$. However, by doing so, we break our working hypothesis that the ϵ_i functions have to be small, as the end of inflation is indeed manifestly violating slow roll.

In spite of this, in essentially all works on slow-roll inflation, $N_{\rm end}^{\rm sr}$ is the value actually used for $N_{\rm end}$. As we will demonstrate in Section 3, this is quite a good approximation because $N_{\rm end}^{\rm sr}$ turns out to be the leading order solution of yet another expansion of the field trajectory valid even when slow roll is violated. Another more intuitive explanation justifying the extrapolation of $\Gamma \simeq \Gamma_{\rm sr}$ to the end of inflation is to remark that when slow-roll is violated inflation cannot be sustained for a long time, typically not more than $\mathcal{O}(1)$ e-fold. As such, one cannot make a larger error than that on $N_{\rm end}$ by using $N_{\rm end}^{\rm sr}$ instead.

In practice, solving $\Gamma_{\rm sr}(N_{\rm end}^{\rm sr}) = \pm\sqrt{2}$ consists in finding the root $\phi_{\rm end}^{\rm sr}$ of the algebraic equation

$$\Gamma_{\rm sr}(\phi_{\rm end}^{\rm sr}) = -\left.\frac{\mathrm{d}\ln V(\phi)}{\mathrm{d}\phi}\right|_{\phi_{\rm end}^{\rm sr}} = \pm\sqrt{2}, \tag{2.13}$$

and injecting it into the slow-roll trajectory of Eq. (2.11), i.e., $N_{\rm end}^{\rm sr} = N_{\rm sr}(\phi_{\rm end}^{\rm sr})$.

The slow-roll approximated trajectory, complemented by its extrapolation to determine the e-fold at which inflation ends, finally gives

$$\Delta N_{\rm sr}(\phi) \equiv N_{\rm sr}(\phi) - N_{\rm end}^{\rm sr} = \int_{\phi}^{\phi_{\rm end}^{\rm sr}} \frac{V(\psi)}{V'(\psi)}\,\mathrm{d}\psi\,, \tag{2.14}$$

where $\phi_{\rm end}^{\rm sr}$ solves Eq. (2.13). The function $\Delta N_{\rm sr}(\phi)$ is the one commonly used to solve the reheating Eq. (1.9). Let us now discuss its accuracy.

2.4 Assessing slow-roll accuracy

In this section, we compare, for various potentials, the slow-roll approximated trajectory $\Delta N_{\rm sr}(\phi)$ defined by Eq. (2.14), to an exact numerical integration of Eq. (2.7) complemented by a root finding algorithm to numerically determine $\phi_{\rm end}$, the solution of $\Gamma(\phi_{\rm end}) = \pm\sqrt{2}$.

As discussed in the previous sections, the errors made by using $\Delta N_{\rm sr}(\phi)$ instead of $\Delta N(\phi)$ come from both the approximation $\Gamma \simeq \Gamma_{\rm sr}$ and $\phi_{\rm end} \simeq \phi_{\rm end}^{\rm sr}$. In order to separate both, let us define a semi-numerical solution, built upon the slow-roll trajectory, based on the exact field value for the end of inflation

$$\Delta N_{\rm sr}^{\rm ee} \equiv N_{\rm sr}(\phi) - N_{\rm sr}(\phi_{\rm end}). \tag{2.15}$$

All these functions take as input the field value ϕ and return some approximated number of e-folds. Once we have (numerically) integrated the field trajectory exactly, we have at our disposal the functions $N(\phi)$, $\phi(N)$ as well as the value of $\phi_{\rm end}$ and $N_{\rm end} = N(\phi_{\rm end})$. From these, we can numerically determine the exact functions $\Delta N(\phi)$ and $\phi(\Delta N)$.

Starting from some initial conditions, at $\phi = \phi_{\rm ini}$ and $N_{\rm ini} = 0$, a first approach would be to compare the exact solution $N(\phi)$ to its slow-roll approximated version $N_{\rm sr}(\phi)$. Equivalently, one could also compare $\phi(N)$ to $\phi^{\rm sr}(N)$. Let us first remark that the field value $\phi_{\rm ini}$ plays no role as, for a given potential, and once on the attractor, the trajectory $\phi(N)$ is universal and always ends in the same manner. Intuitively, one expects $N_{\rm sr}(\phi) - N(\phi)$, as well as $\phi^{\rm sr}(N) - \phi(N)$, to be small deep in slow-roll while growing towards the end of inflation and this is exactly what happens. However, we have chosen not to show these trajectories in the following. Indeed, as discussed at length in the introduction, the observable quantity entering the reheating equation is $\Delta N = N(\phi) - N_{\rm end}$, in reference to the end of inflation. As such, any errors damaging the actual value of $N_{\rm end}$, as the ones building up close to the end of inflation, will be necessarily folded into *all the values* of ΔN, even if $N(\phi)$ and $N_{\rm sr}(\phi)$ match well in those regions. Hence, it is actually much more informative to compare $\Delta N_{\rm sr}(\phi)$, $\Delta N_{\rm sr}^{\rm ee}(\phi)$ to the exact $\Delta N(\phi)$, all of these quantities being sensitive to the end of inflation.

Last but not least, the values taken by ϕ are also not very much informative as observable predictions are mostly sensitive to e-fold numbers. Knowing the exact trajectory $\phi(\Delta N)$, we can easily trade ϕ for ΔN and discuss all error made in terms of the latter quantity. This is relevant because the prototypical value of $\Delta N \simeq N_0 \simeq -61.5$ and minimizing errors is particularly important around these figures rather than towards the end of inflation, or, much earlier.

In Fig. 1, we have therefore represented, for various models, as a function of $\Delta N(\phi)$, the absolute errors in the number of e-folds made by using the slow-roll approximated trajectories instead of the exact one. The red curve in these plots shows $\Delta N_{\rm sr}(\phi) - \Delta N(\phi)$ whereas the blue curve is for the semi-analytical trajectory $\Delta N_{\rm sr}^{\rm ee}(\phi) - \Delta N(\phi)$. The differences between the blue and red curves are thus coming from the uncertainties in determining the field value at which inflation ends.

The six models considered in Fig. 1 are a few representative of the ones discussed in the *Encyclopædia Inflationaris* paper of Ref. [14]. We have picked up a quadratic large field inflation model (LFI_2), having a potential $V(\phi) \propto \phi^2$, a quartic small field inflation model ($\mathrm{SFI}_{4\mathrm{l}}$) with $V(\phi) \propto 1 - (\phi/\mu)^4$ where the vacuum expectation value $\mu = 10$ is super-Planckian, Starobinsky Inflation (SI) having $V^{1/2}(\phi) \propto 1 - e^{-\sqrt{2/3}\phi}$, a quadratic T-model inflation (TMI) with $V(\phi) \propto \tanh^2(\phi/\sqrt{6})$, an exponential supersymmetric inflation model (ESI_1) with $V(\phi) \propto 1 - e^{-\phi}$ and a pseudo-natural inflation model (PSNI) having a potential $V(\phi) \propto 1 + \alpha \ln(\cos\phi)$ with $\alpha = 1/10$. Not all of these models are compatible with current cosmological data, for instance LFI_2 is strongly disfavoured whereas PSNI has its parameters purposely chosen to violate slow-roll ($\epsilon_2 \simeq 0.2$ during inflation). The small field scenario is a model which is compatible with the data whereas SI, TMI and ESI_1 are different incarnations of the so-called plateau-type models and belong to most favoured scenarios [19].

Let us remark in Fig. 1 that, for all models, the errors generated by the slow-roll trajectory of Eq. (2.11) grow with the number of e-folds before the end of inflation. This growth is precisely due to the small terms omitted by making the assumption $\Gamma \simeq \Gamma_{\rm sr}$ and confirms that $\Gamma_{\rm sr}$ is slightly off-track the true slow rolling attractor velocity. One can also notice that the error jumps quite fast close to $\Delta N = 0$ whereas, up to one model (PSNI), it increases like a logarithm at larger values of $|\Delta N|$. This is due to the fact that slow-roll is most violated towards the end of inflation and the assumption $\Gamma \simeq \Gamma_{\rm sr}$ is quite wrong in these regions. For PSNI (lower right panel), the errors seem to increase linearly with $|\Delta N|$, as opposed to a logarithm-like growth. The reason being that, as aforementioned, it is far from slow roll also during inflation ($\epsilon_2 = 0.2$).

Finally, these plots confirm that for the fiducial value $\Delta N \simeq N_0$, the typical errors on the trajectory are $\mathcal{O}(1)$ e-folds. Only for the extreme slow-roll violating model PSNI, one gets a larger, but still reasonable error.

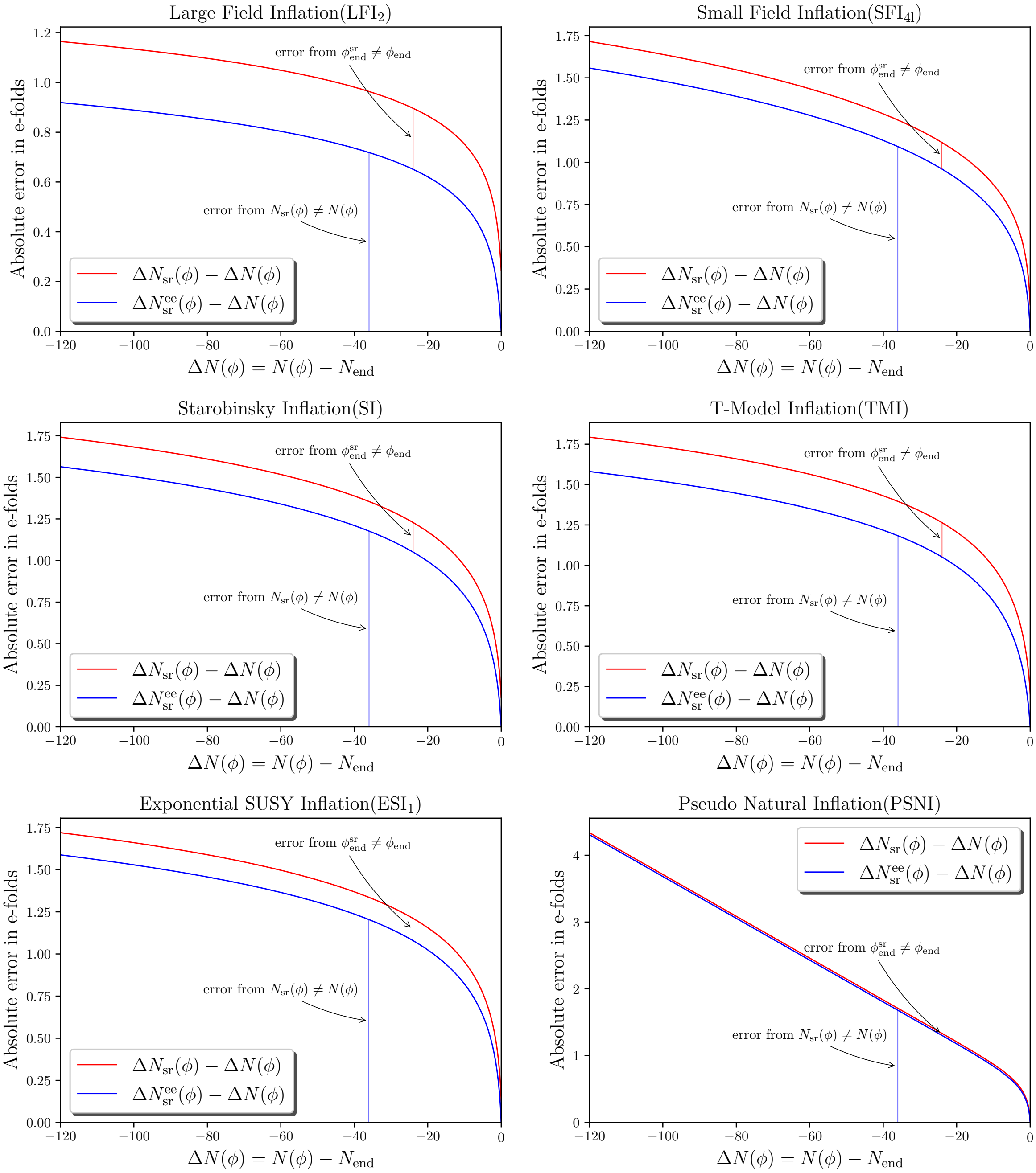

Figure 1. Absolute error, in e-folds, of the slow-roll approximated trajectory (in red) with respect to the exact value of $\Delta N(\phi)$ for various prototypical models of inflation. The blue curve shows $\Delta N^{\mathrm{ee}}_{\mathrm{sr}}(\phi) - \Delta N(\phi)$ where $\Delta N^{\mathrm{ee}}_{\mathrm{sr}} = N_{\mathrm{sr}}(\phi) - N_{\mathrm{sr}}(\phi_{\mathrm{end}})$, ϕ_{end} being the *exact* field value at which inflation stops. The differences between the red and blue curves are the errors induced by using $\phi^{\mathrm{sr}}_{\mathrm{end}}$ instead of ϕ_{end} (see text). Let us notice that the Pseudo Natural Inflationary model (lower right) is an extreme case as it has its parameters purposely chosen to be in a slow-roll violating regime (incompatible with current data).

3 Correcting slow-roll

In this section, we address the main source of error eroding the traditional slow-roll trajectory: Γ_{sr} being slightly off-track the attractor solution.

3.1 Integral constraints

Let us first show that, even though the slow-roll trajectory is not right on the attractor, the shift with respect to the exact solution is actually bounded. One can define the absolute error

$$\mathcal{E} \equiv \Gamma - \Gamma_{\mathrm{sr}}, \tag{3.1}$$

which can be viewed as a function $\mathcal{E}(\phi)$ by formally making use of the exact field trajectory for $\Gamma[N(\phi)]$. From the definition of Γ in Eq. (2.5) (still in Planck units) one can rewrite Eq. (2.7) in terms of ϕ as

$$\frac{2\Gamma}{6-\Gamma^2}\frac{\mathrm{d}\Gamma}{\mathrm{d}\phi} + \Gamma = \Gamma_{\mathrm{sr}}. \tag{3.2}$$

This equation can actually be integrated by separating variables and isolating $\mathcal{E}(\phi)$ as

$$\int_{\Gamma}^{\Gamma_{\mathrm{end}}} \frac{2\gamma}{6-\gamma^2}\,\mathrm{d}\gamma = -\int_{\phi}^{\phi_{\mathrm{end}}} \mathcal{E}(\psi)\,\mathrm{d}\psi\,. \tag{3.3}$$

The left-hand side can be integrated exactly and, using $\Gamma_{\mathrm{end}}^2 = 2$, one gets the integral constraint

$$\int_{\phi}^{\phi_{\mathrm{end}}} \mathcal{E}(\psi)\,\mathrm{d}\psi = \ln\left[\frac{4}{6-\Gamma^2(\phi)}\right]. \tag{3.4}$$

In the slow roll regime $\Gamma^2 \ll 1$ and the integrated error made between ϕ and ϕ_{end} is $\ln(2/3) \simeq -0.4$. Notice the negative sign, which shows that, for $\phi_{\mathrm{end}} > \phi$, one has $\Gamma_{\mathrm{sr}} \gtrsim \Gamma > 0$ and the approximated trajectory is slightly advanced compared to the exact one (see also Fig. 1).

We can also derive a second integral constraint by integrating $\mathcal{E}(N)$ with respect to the number of e-fold. Starting again from Eq. (2.7) and separating the variables Γ and N, one has

$$\int_{\Gamma}^{\Gamma_{\mathrm{end}}} \frac{2}{6-\gamma^2}\,\mathrm{d}\gamma = -\int_{N}^{N_{\mathrm{end}}} \mathcal{E}(n)\,\mathrm{d}n\,. \tag{3.5}$$

Again, the left-hand side can be integrated exactly while the right hand side can be expressed in terms of ϕ by using Eq. (2.5). One obtains another integral constraint

$$\int_{\phi}^{\phi_{\mathrm{end}}} \frac{\mathcal{E}(\psi)}{\Gamma(\psi)}\,\mathrm{d}\psi = \frac{1}{\sqrt{6}}\ln\left[\left(2\mp\sqrt{3}\right)\frac{\sqrt{6}+\Gamma(\phi)}{\sqrt{6}-\Gamma(\phi)}\right], \tag{3.6}$$

where $\mathcal{E}/\Gamma$ is the *relative* error between the slow-roll and exact trajectory. The $\pm$ sign is for $\Gamma_{\mathrm{end}} = \pm\sqrt{2}$, depending on which direction inflation proceeds. Provided $|\Gamma(\phi)| \ll 1$, the relative integrated error is bounded and reads $\ln\left(2\mp\sqrt{3}\right)/\sqrt{6} \simeq \mp 0.53$. We recover that, for $\phi_{\mathrm{end}} > \phi$, $\mathcal{E} < 0$ and $\Gamma_{\mathrm{sr}} \gtrsim \Gamma > 0$.

Both Eqs. (3.4) and (3.6) are finite and shows that both the absolute error $\mathcal{E}$ and the relative error $\mathcal{E}/\Gamma$ are under control, deep in the slow roll regime as well as at the end of inflation when Γ approaches Γ_{end}. This suggests using either $\mathcal{E}(\phi)$ or $\mathcal{E}(\phi)/\Gamma(\phi)$ as a small parameter against which the exact solution of Eq. (2.7) can be expanded.

3.2 New expansion for the field trajectory

From the exact field velocity $\Gamma(\phi)$, the true number of e-fold is given, up to a constant, by

$$N(\phi) = \int^{\phi} \frac{1}{\Gamma(\psi)}\,\mathrm{d}\psi\,. \tag{3.7}$$

Instead, the traditional slow-roll approximation replaces it with Eq. (2.11). Let us use Eq. (3.7) to express the exact field trajectory as

$$\Delta N(\phi) = N(\phi) - N_{\mathrm{end}} = -\int_{\phi}^{\phi_{\mathrm{end}}} \frac{1}{\Gamma_{\mathrm{sr}}(\psi)}\frac{\Gamma_{\mathrm{sr}}(\psi)}{\Gamma(\psi)}\,\mathrm{d}\psi\,, \tag{3.8}$$

where we have artificially introduced the known function $\Gamma_{\mathrm{sr}}(\phi)$ defined in Eq. (2.10). From the definition of $\mathcal{E}$ in Eq. (3.1), one has

$$\frac{\Gamma_{\mathrm{sr}}}{\Gamma} = 1 - \frac{\mathcal{E}}{\Gamma}, \tag{3.9}$$

which can be plugged into Eq. (3.8) to get

$$\Delta N(\phi) = -\int_{\phi}^{\phi_{\mathrm{end}}} \frac{1}{\Gamma_{\mathrm{sr}}(\psi)}\,\mathrm{d}\psi + \int_{\phi}^{\phi_{\mathrm{end}}} \frac{\mathcal{E}(\psi)}{\Gamma^2(\psi)}\frac{\Gamma(\psi)}{\Gamma_{\mathrm{sr}}(\psi)}\,\mathrm{d}\psi\,, \tag{3.10}$$

the first term giving back the traditional slow-roll approximation. The second term can be further expanded by remarking that

$$\frac{\Gamma}{\Gamma_{\mathrm{sr}}} = \frac{1}{1-\dfrac{\mathcal{E}}{\Gamma}} = 1 + \sum_{k=1}^{\infty}\left(\frac{\mathcal{E}}{\Gamma}\right)^k, \tag{3.11}$$

giving a new and exact expansion for the field trajectory

$$\Delta N(\phi) = \Delta N_{\mathrm{sr}}^{\mathrm{ee}}(\phi) + \int_{\phi}^{\phi_{\mathrm{end}}} \frac{\mathcal{E}(\psi)}{\Gamma^2(\psi)} + \sum_{k=2}^{\infty}\int_{\phi}^{\phi_{\mathrm{end}}} \frac{1}{\Gamma(\psi)}\left[\frac{\mathcal{E}(\psi)}{\Gamma(\psi)}\right]^k \mathrm{d}\psi\,. \tag{3.12}$$

Let us notice the first term, which is $\Delta N_{\mathrm{sr}}^{\mathrm{ee}}$ as the field value ϕ_{end} here has to be the exact one. Quite importantly, this expansion is *not* based on the usual slow-roll expansion, one does not need to assume $|\Gamma| \ll 1$. Instead, the "small parameter" is the relative error functional, $\mathcal{E}/\Gamma$, which is ensured to be under control thanks to the integral constraints derived earlier. The benefit of having expanded the exact trajectory as in Eq. (3.12) is that, as we show in the next section, the second term, which acts as a first correction, is exactly calculable.

Strictly speaking, the expansion of Eq. (3.11) is converging only if the relative error $|\mathcal{E}/\Gamma| < 1$. Although this is ensured for most of the inflationary trajectory, in slow roll, it may exceed unity very close to the end of inflation for $|\Gamma_{\mathrm{sr}}| > 2|\Gamma| \simeq 2\sqrt{2}$. However, in order to satisfy the integral constraint of Eq. (3.6), the field domain over which this happens must be small (in Planck units). Similarly, Eq. (3.4) gives a constraint on the absolute error $\mathcal{E}$ over time, this one cannot not be of order unity for more than a fraction of an e-fold. Although these cases are not of immediate interest when considering $\Delta N \simeq N_0$, it is interesting to remark that for $|\mathcal{E}/\Gamma| > 1$, one has $|\mathcal{E}/\Gamma_{\mathrm{sr}}| < 1$ and another expansion can be performed

$$\frac{\Gamma_{\mathrm{sr}}}{\Gamma} = \frac{1}{1+\dfrac{\mathcal{E}}{\Gamma_{\mathrm{sr}}}} = 1 + \sum_{k=1}^{\infty}(-1)^k\left(\frac{\mathcal{E}}{\Gamma_{\mathrm{sr}}}\right)^k. \tag{3.13}$$

Plugging this expression into Eq. (3.8), one gets another exact expansion

$$\Delta N(\phi) = \Delta N_{\mathrm{sr}}^{\mathrm{ee}}(\phi) + \int_{\phi}^{\phi_{\mathrm{end}}} \frac{\mathcal{E}(\psi)}{\Gamma_{\mathrm{sr}}^2(\psi)} - \sum_{k=2}^{\infty} \int_{\phi}^{\phi_{\mathrm{end}}} \frac{(-1)^k}{\Gamma_{\mathrm{sr}}(\psi)} \left[\frac{\mathcal{E}(\psi)}{\Gamma_{\mathrm{sr}}(\psi)}\right]^k \mathrm{d}\psi\,, \tag{3.14}$$

which shows that, up to the field value at which inflation ends, the usual slow-roll approximated trajectory $\Delta N_{\mathrm{sr}}^{\mathrm{ee}}$ remains the leading order term.

3.3 Velocity correction

Let us now assume that we are in the field domain for which $|\mathcal{E}/\Gamma| < 1$. From Eq. (3.2), one has, exactly

$$\frac{\mathcal{E}(\phi)}{\Gamma^2(\phi)} = -\frac{2}{\Gamma(\psi)\left[6-\Gamma^2(\psi)\right]} \frac{\mathrm{d}\Gamma}{\mathrm{d}\phi}\,, \tag{3.15}$$

such that

$$\int_{\phi}^{\phi_{\mathrm{end}}} \frac{\mathcal{E}(\psi)}{\Gamma^2(\psi)} \mathrm{d}\psi = \frac{1}{6}\ln\left[\frac{\Gamma^2(\psi)}{6-\Gamma^2(\psi)}\right] - \frac{1}{6}\ln\left[\frac{\Gamma_{\mathrm{end}}^2}{6-\Gamma_{\mathrm{end}}^2}\right] = \frac{1}{6}\ln\left[\frac{2\Gamma^2(\psi)}{6-\Gamma^2(\psi)}\right], \tag{3.16}$$

where use has been made of $\Gamma_{\mathrm{end}}^2 = 2$. As such, the first correction appearing in the expansion of Eq. (3.12) is a simple velocity correction. Even in slow-roll, for $|\Gamma| \ll 1$, we see that this term matters. As a matter of fact, it grows logarithmically when Γ becomes small, and it cancels most of the errors associated with $\Delta N - \Delta N_{\mathrm{sr}}^{\mathrm{ee}}$.

We can also rewrite Eq. (3.12) as

$$\Delta N(\phi) = \Delta N_{\mathrm{sr}}(\phi) + \frac{1}{6}\ln\left[\frac{2\Gamma^2(\phi)}{6-\Gamma^2(\phi)}\right] + \left[\Delta N_{\mathrm{sr}}^{\mathrm{ee}}(\phi) - \Delta N_{\mathrm{sr}}(\phi)\right] + \sum_{k=2}^{\infty} \int_{\phi}^{\phi_{\mathrm{end}}} \frac{1}{\Gamma(\psi)} \left[\frac{\mathcal{E}(\psi)}{\Gamma(\psi)}\right]^k \mathrm{d}\psi\,, \tag{3.17}$$

to render explicit the deviations with respect to $\Delta N_{\mathrm{sr}}(\phi)$. The third term appears because ΔN_{sr} assumes inflation to end at $\phi_{\mathrm{end}}^{\mathrm{sr}}$ instead of the exact value ϕ_{end}. We will discuss this issue in Section 4.

An additional issue with Eq. (3.17) is that, in principle, one does not know $\Gamma(\phi)$. However, we are interested in $\Delta N(\phi)$ far from the end of inflation, where slow roll is verified. As such, it is perfectly justified to evaluate $\Gamma(\phi) \simeq \Gamma_{\mathrm{sr}}(\phi)$ within the second term of Eq. (3.17). If a higher precision is needed, it is always possible to account for higher-derivative perturbative corrections by determining Γ in terms of Γ_{sr} [19, 92]. In fact, it is interesting to compare our velocity refinement, the second term of Eq. (3.17), to these higher-derivative corrections. As shown in Ref. [93], including the next-to-leading order term to determine $\Gamma(\Gamma_{\mathrm{sr}}, \Gamma_{\mathrm{sr}}')$ yields a correction to $N_{\mathrm{sr}}(\phi)$ in $\ln\left[\Gamma_{\mathrm{sr}}^2(\phi)\right]/6$. This can be compared to first term in the right-hand side of Eq. (3.16), i.e., $\ln\left\{\Gamma^2(\phi)/[6-\Gamma(\phi)^2]\right\}/6$. Deep in slow roll, using $|\Gamma| \simeq |\Gamma_{\mathrm{sr}}| \ll 1$, it approximates to $\ln\left[\Gamma_{\mathrm{sr}}^2(\phi)/6\right]/6$ and this improves the next-to-leading order higher-derivative correction by a constant shift of $-\ln(6)/6 \simeq -0.3$ e-fold. Let us also stress that, as opposed to perturbative higher-order terms, Eq. (3.16) is exact and this is why we can safely incorporate into the velocity correction some effects coming from the end of inflation (the terms involving Γ_{end}).

In order to check the accuracy of Eq. (3.17), we have plotted in Fig. 2 the absolute error $\Delta N_{\mathrm{sr}}^{\mathrm{vc}}(\phi) - \Delta N(\phi)$ (blue curve) where

$$\Delta N_{\mathrm{sr}}^{\mathrm{vc}}(\phi) \equiv \Delta N_{\mathrm{sr}}(\phi) + \frac{1}{6}\ln\left[\frac{2\Gamma_{\mathrm{sr}}^2(\phi)}{6-\Gamma_{\mathrm{sr}}^2(\phi)}\right]. \tag{3.18}$$

Let us stress that we have traded Γ for $\Gamma_{\rm sr}$ in this expression. Compared to the traditional slow-roll trajectory (red), it is evident that the velocity corrections erase the logarithmic error growth with respect to ΔN by a factor $\mathcal{O}(10)$. In view of such a success, one may be tempted in trying to evaluate the higher order terms of Eq. (3.17) similarly. However, because they involve powers of $\mathcal{E}/\Gamma$, using Eq. (3.15) does not allow for an exact integration, even though some parts can still be estimated. We have also tried to calculate exactly these terms starting from Eq. (2.7) and this has allowed us to derive a new exact formula, presented in Appendix A, but which would require the knowledge of $V(N)$, which is not usually the case (see, however, Section 4.2).

Moreover, one can see from Fig. 2 that the remaining error (blue curve) $\Delta N_{\rm sr}^{\rm vc}(\phi) - \Delta N(\phi)$ is almost stationary (with respect to ΔN) and only driven by the higher order terms of Eq. (3.17). One of them, the third one, encodes the inaccuracies due to the value of $\phi_{\rm end}^{\rm sr}$. We now turn to this question.

4 Pinpointing the end of inflation

As explained in Section 2.3, the uncertainties associated with $\phi_{\rm end}^{\rm sr}$ come from the trading between Γ and $\Gamma_{\rm sr}$ at the end of inflation, when slow-roll is manifestly violated. A first approach may be to use Eq. (3.9) close to the end of inflation and making use of Eq. (3.15) while replacing Γ by $\Gamma_{\rm sr}$ in the derivative. However, we are now in a regime in which Γ is not necessarily close to $\Gamma_{\rm sr}$ and $|\mathcal{E}/\Gamma|$ could also exceed unity. We have checked that the relative error $\mathcal{E}/\Gamma$ indeed exceeds unity at the end of inflation for two of the tested models: ESI_1 and PSNI. A similar approach has been discussed in Refs. [94, 95] and it has been shown to reduce the error in determining the end of inflation down to 5% for some specific potentials (LFI_2 and TMI-like, which both satisfy $|\mathcal{E}/\Gamma| < 1$). In the following, we will present other methods, performing only slightly better (2% uncertainties on $\phi_{\rm end}$) but designed to be robust for all slow-roll models.

4.1 Constrained extrapolations

For a given potential, one has a perfect knowledge of $\Gamma_{\rm sr}(\phi)$ and $\phi_{\rm end}^{\rm sr}$, and we would like to have an accurate determination of $\Gamma(\phi)$ close to $\phi_{\rm end}$. As such, we could circumvent the determination of $\Gamma(\phi)$ by trying to approximate instead the integral constraints of the error function $\mathcal{E}(\phi)$ towards the end of inflation.

For instance, using a trapezoidal approximation for the integrals, one can approximate the first constraint Eq. (3.4) at the end of inflation as

$$\ln\left[\frac{4}{6-\Gamma^2\left(\phi_{\rm end}^{\rm sr}\right)}\right] = \int_{\phi_{\rm end}^{\rm sr}}^{\phi_{\rm end}} \mathcal{E}(\psi)\,\mathrm{d}\psi \approx \frac{1}{2}\left[\mathcal{E}(\phi_{\rm end}) + \mathcal{E}(\phi_{\rm end}^{\rm sr})\right](\phi_{\rm end} - \phi_{\rm end}^{\rm sr}). \tag{4.1}$$

Similarly, one may approximate the second constraint Eq. (3.6), as

$$\frac{1}{\sqrt{6}}\ln\left[\left(2\mp\sqrt{3}\right)\frac{\sqrt{6}+\Gamma\left(\phi_{\rm end}^{\rm sr}\right)}{\sqrt{6}-\Gamma\left(\phi_{\rm end}^{\rm sr}\right)}\right] = \int_{\phi_{\rm end}^{\rm sr}}^{\phi_{\rm end}} \frac{\mathcal{E}(\psi)}{\Gamma(\psi)}\,\mathrm{d}\psi \approx \frac{1}{2}\left[\frac{\mathcal{E}(\phi_{\rm end})}{\Gamma(\phi_{\rm end})} + \frac{\mathcal{E}(\phi_{\rm end}^{\rm sr})}{\Gamma\left(\phi_{\rm end}^{\rm sr}\right)}\right](\phi_{\rm end} - \phi_{\rm end}^{\rm sr}). \tag{4.2}$$

Under these approximations, Eqs. (4.1) and (4.2) only involve two unknown and independent variables $\phi_{\rm end}$ and $\Gamma(\phi_{\rm end}^{\rm sr})$. Indeed, the error function in both points can be written as

$$\mathcal{E}(\phi_{\rm end}) = \Gamma_{\rm end} - \Gamma_{\rm sr}(\phi_{\rm end}), \quad \text{and } \mathcal{E}(\phi_{\rm end}^{\rm sr}) = \Gamma(\phi_{\rm end}^{\rm sr}) - \Gamma_{\rm end}, \tag{4.3}$$

where, as before, $\Gamma_{\rm end} = \pm\sqrt{2}$. This algebraic system can be solved numerically to obtain an estimate of the field value at the end of inflation $\phi_{\rm end}^{\pi}$. Let us stress that, according to the previous discussion, the domain in which these equations are solved requires $|\Gamma(\phi_{\rm end}^{\rm sr})| < |\Gamma_{\rm end}|$ and, either $\phi_{\rm end}^{\pi} < \phi_{\rm end}^{\rm sr}$ or $\phi_{\rm end}^{\pi} > \phi_{\rm end}^{\rm sr}$, depending on whether inflation proceeds at decreasing or increasing field values, respectively.

Let us notice that the method could also be accommodated with other functional shapes to model $\mathcal{E}(\phi)$ in the constraint integrals. We have tested some power-law and exponential extrapolations, but they do not perform better than the simple trapezoidal rule presented here. In the next section, we discuss a similar method anchored on a family of exact field trajectories.

4.2 Matching to Mukhanov inflation

Mukhanov inflation is one of the very few inflationary models for which the exact field trajectory is analytically known, and, the only one which exhibits a graceful exit [96].

Without giving details, the potential is parametrized by two constants α and β and reads, in Planck units [14]

$$V_{\rm m}(\phi) = M^4 \left[1 - \frac{\beta}{2\left(1+\frac{2-\alpha}{2}\frac{\phi}{\sqrt{3\beta}}\right)^{\frac{2\alpha}{2-\alpha}}}\right] \exp\left\{\frac{3\beta}{1-\alpha}\left[\left(1+\frac{2-\alpha}{2}\frac{\phi}{\sqrt{3\beta}}\right)^{\frac{2(1-\alpha)}{2-\alpha}} - 1\right]\right\}. \tag{4.4}$$

The expression for $\Gamma(\phi)$ is analytically known and reads

$$\Gamma_{\rm m}^2(\phi) = \frac{3\beta}{\left(1+\frac{2-\alpha}{2}\frac{\phi}{\sqrt{3\beta}}\right)^{\frac{2\alpha}{2-\alpha}}}. \tag{4.5}$$

The field value at which Mukhanov inflation ends is obtained by solving $\Gamma_{\rm m}^2(\phi_{\rm end}^{\rm m}) = 2$ and reads[2]

$$\phi_{\rm end}^{\rm m} = \frac{2\sqrt{3\beta}}{2-\alpha}\left[\left(\frac{3\beta}{2}\right)^{\frac{2-\alpha}{2\alpha}} - 1\right]. \tag{4.6}$$

These functions being exact solution of Eq. (2.7), they automatically satisfy all the integral constraints discussed in Section 3.1.

The present problem is to estimate $\phi_{\rm end}$ knowing only $\Gamma_{\rm sr}(\phi)$ and one can use Mukhanov inflation as a proxy. For instance, one can determine the value of α and β such that $\Gamma_{\rm sr}^{\rm m} \simeq \Gamma_{\rm sr}$. If the slow-roll trajectories of Mukhanov inflation and of the model under scrutiny are close, then should also be their respective exact trajectories. As such, one can use $\phi_{\rm end}^{\rm m}$ as an approximation of the unknown $\phi_{\rm end}$.

In order to determine the two parameters α and β, one needs two equations, which cannot be provided by the integral constraints of Section 3.1 as they are already verified. Let us match the first and second derivatives of the potential's logarithm towards the end of inflation by imposing

$$\Gamma_{\rm sr}^{\rm m}(\phi_{\rm end}^{\rm sr}) = \Gamma_{\rm end}, \qquad \left.\frac{{\rm d}\Gamma_{\rm sr}^{\rm m}}{{\rm d}\phi}\right|_{\phi_{\rm end}^{\rm sr}} = \left.\frac{{\rm d}\Gamma_{\rm sr}}{{\rm d}\phi}\right|_{\phi_{\rm end}^{\rm sr}}. \tag{4.7}$$

[2] Another solution exists for $\alpha > 2$ as the potential develops a maximum located at $\phi_{\max} = 2\sqrt{3\beta}/(\alpha-2)$ and there is a symmetry $\phi \to 2\phi_{\max} - \phi$ in these cases.

	LFI_2		SFI_{4l}		SI		TMI		ESI_1		PSNI	
ϕ_{end}	1.009	0%	9.657	0%	0.615	0%	0.839	0%	0.271	0%	1.564	0%
$\phi_{\mathrm{end}}^{\mathrm{sr}}$	1.414	40%	9.361	3.1%	0.940	53%	1.208	44%	0.535	97%	1.478	5%
$\phi_{\mathrm{end}}^{\pi}$	0.984	2.5%	9.661	0.04%	0.607	1.3%	0.826	1.5%	0.273	0.7%	–	–
$\phi_{\mathrm{end}}^{\mathrm{m}}$	0.986	2.3%	9.678	0.2%	0.594	3.4%	0.825	1.7%	0.238	12%	1.604	2%

Table 1. Comparison of various approximations to determine the field value at which inflation ends. The exact value is ϕ_{end}, the slow-roll approximation gives $\phi_{\mathrm{end}}^{\mathrm{sr}}$, the trapezoid error function extrapolation yields $\phi_{\mathrm{end}}^{\pi}$, and, matching Mukhanov inflation at $\phi_{\mathrm{end}}^{\mathrm{sr}}$ gives the value $\phi_{\mathrm{end}}^{\mathrm{m}}$. The number quoted in percent is the relative error in reference to the exact value ϕ_{end}. The gain in precision on ϕ_{end} by the methods presented here is about an order of magnitude compared to slow-roll.

Here, $\phi_{\mathrm{end}}^{\mathrm{sr}}$ refers to the slow-roll approximated value, precisely obtained by solving $\Gamma_{\mathrm{sr}}(\phi_{\mathrm{end}}^{\mathrm{sr}}) = \Gamma_{\mathrm{end}}$, while the slow-roll velocity $\Gamma_{\mathrm{sr}}^{\mathrm{m}}(\phi)$ of Mukhanov inflation has still to be determined. From Eq. (4.4), one has

$$\Gamma_{\mathrm{sr}}^{\mathrm{m}}(\phi) \equiv -\frac{\mathrm{d}\ln V_{\mathrm{m}}}{\mathrm{d}\phi} = -\sqrt{\frac{\beta}{3}}\frac{x^{\alpha}\left(\alpha+6x\right)-3\beta x}{x^{1+\frac{\alpha}{2}}\left(2x^{\alpha}-\beta\right)}\,, \tag{4.8}$$

where

$$x(\phi) \equiv \left(1+\frac{2-\alpha}{2}\frac{\phi}{\sqrt{3\beta}}\right)^{\frac{2}{2-\alpha}}. \tag{4.9}$$

Taking the derivative of Eq. (4.8) with respect to ϕ gives

$$\frac{\mathrm{d}\Gamma_{\mathrm{sr}}^{\mathrm{m}}}{\mathrm{d}\phi} = \frac{2\left(\alpha+2\right)\left(\alpha+6x\right)x^{2\alpha}-\beta\left[\alpha\left(2-\alpha\right)+12x\left(\alpha+1\right)\right]x^{\alpha}+3\alpha\beta^{2}x}{6x^{2}\left(2x^{\alpha}-\beta\right)^{2}}\,. \tag{4.10}$$

Let us mention that the slow-roll approximated functions of Eqs. (4.8) and (4.10) are also related to the so-called potential slow-roll parameters $\epsilon_{1V} = \left(\Gamma_{\mathrm{sr}}^{\mathrm{m}}\right)^2/2$ and $\epsilon_{2V} = 2\,\mathrm{d}\Gamma_{\mathrm{sr}}^{\mathrm{m}}/\mathrm{d}\phi$.

For a given inflationary potential $V(\phi)$, plugging Eqs. (4.8) and (4.10) into Eq. (4.7) gives a set of two algebraic equations for α and β that has to be solved numerically. Once the value of α and β are determined, the exact field value at which Mukhanov inflation ends is given by Eq. (4.6) and this will be taken as the estimator of the unknown ϕ_{end}.

4.3 End point correction

In Table 1, we give the exact numerical value of ϕ_{end}, the slow-roll approximated value $\phi_{\mathrm{end}}^{\mathrm{sr}}$, the trapezoid-approximated value $\phi_{\mathrm{end}}^{\pi}$ and the Mukhanov-approximated $\phi_{\mathrm{end}}^{\mathrm{m}}$, for the six inflationary models presented in Section 2.4. For the quite extreme case PSNI, we have not reported the value of $\phi_{\mathrm{end}}^{\pi}$ as we have found more than one numerical solution for Eqs. (4.1) and (4.2) thereby preventing an easy determination of the best estimator. Let us notice that the value of $\phi_{\mathrm{end}}^{\mathrm{m}}$, even if close to the exact one, lies in a domain for which the PSNI potential is not defined ($\phi > \pi/2$) and, as such, it is certainly not really useful. Let us stress, again, that numerically solving algebraic equations, such as Eqs. (4.1) and (4.2), or, Eq. (4.7), is orders of magnitude faster than numerically integrating Eq. (2.7) all along inflation. As can be seen in this table, the relative error in reference to ϕ_{end} is reduced by an order of magnitude using either the trapezoidal approximation or the Mukhanov inflation matching method instead of the traditional slow-roll value.

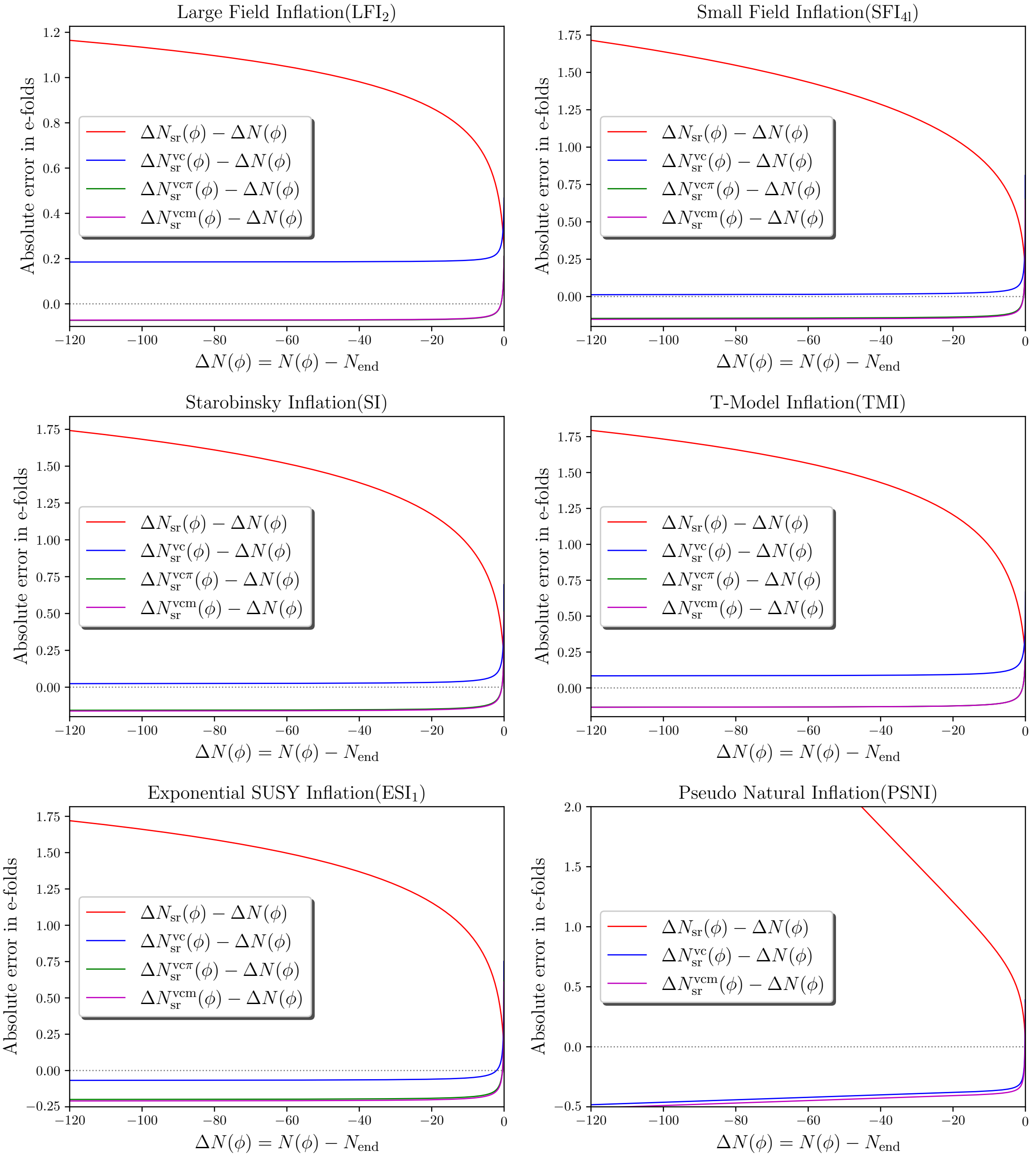

Figure 2. Absolute error, in e-folds, of the velocity-corrected trajectory $\Delta N_{\rm sr}^{\rm vc} - \Delta N$ (blue curve), of the velocity plus end-point corrected trajectories $\Delta N_{\rm sr}^{\rm vcm} - \Delta N$ (green curve) and $\Delta N_{\rm sr}^{\rm vc\pi} - \Delta N$ (magenta curve), with respect to the exact value of $\Delta N(\phi)$ for various prototypical models of inflation. The red curve is the error associated with the traditional slow-roll approximation, same as in Fig. 1.

One can now define an additional correction to slow roll, including both the velocity correction of Section 3.3 and a better estimation of the end point field value, as

$$\begin{aligned}\Delta N_{\mathrm{sr}}^{\mathrm{vc}\pi} &\equiv N_{\mathrm{sr}}(\phi) - N_{\mathrm{sr}}(\phi_{\mathrm{end}}^{\pi}) + \frac{1}{6}\ln\left[\frac{2\Gamma_{\mathrm{sr}}^2(\phi)}{6-\Gamma_{\mathrm{sr}}^2(\phi)}\right] \\ &= \Delta N_{\mathrm{sr}}(\phi) + \frac{1}{6}\ln\left[\frac{2\Gamma_{\mathrm{sr}}^2(\phi)}{6-\Gamma_{\mathrm{sr}}^2(\phi)}\right] + N_{\mathrm{sr}}(\phi_{\mathrm{end}}^{\mathrm{sr}}) - N_{\mathrm{sr}}(\phi_{\mathrm{end}}^{\pi}),\end{aligned} \tag{4.11}$$

and a similar expression for the Mukhanov-approximated value

$$\Delta N_{\mathrm{sr}}^{\mathrm{vcm}} = \Delta N_{\mathrm{sr}}(\phi) + \frac{1}{6}\ln\left[\frac{2\Gamma_{\mathrm{sr}}^2(\phi)}{6-\Gamma_{\mathrm{sr}}^2(\phi)}\right] + N_{\mathrm{sr}}(\phi_{\mathrm{end}}^{\mathrm{sr}}) - N_{\mathrm{sr}}(\phi_{\mathrm{end}}^{\mathrm{m}}). \tag{4.12}$$

The performance of $\Delta N_{\mathrm{sr}}^{\mathrm{vc}\pi}(\phi)$ and $\Delta N_{\mathrm{sr}}^{\mathrm{vcm}}(\phi)$ have been represented in Fig. 2, as green and magenta curves, compared to the traditional slow roll ΔN_{sr} (red) and to the velocity corrected trajectory $\Delta N_{\mathrm{sr}}^{\mathrm{vc}}$ (blue). There is some improvement for LFI_2 while for the other model, using a more accurate field value for the end of inflation produces a slight overshoot for the total correction. As can be checked in Eq. (3.17), the next term ($k = 2$) in the expansion has an opposite sign to the third one, $\Delta N_{\mathrm{sr}}^{\mathrm{ee}} - \Delta N_{\mathrm{sr}}$, which is the error induced by using $\phi_{\mathrm{end}}^{\mathrm{sr}}$ instead ϕ_{end}. As such, they somehow compensate and, for some models, the best option is to keep the simple velocity correction with $\phi_{\mathrm{end}}^{\mathrm{sr}}$ in the trajectory. Notice, however, that a more accurate value for ϕ_{end} is always beneficial for estimating ρ_{end} while the small overshooting is never large enough to spoil the velocity correction.

5 Conclusion

In this work, we have proposed new methods to improve the analytical observable predictions of the slow-roll single field models of inflation. Complementing most of the works in the literature that have been focused on the generation of cosmological perturbations during inflation, we have been focused here on a quite neglected aspect which concerns the accuracy at which the background field trajectory can be determined. As explained in the introduction, determining with precision the relation $\Delta N(\phi)$ is crucial to correctly map wavenumbers today to wavenumbers during inflation. Moreover, because the reheating era lies in between the standard hot Big-Bang model eras and Cosmic Inflation, any uncertainties on $\Delta N(\phi)$ will bias any inference made onto the kinematics of the reheating era.

One of the main results of this work is the exact expansion of Eq. (3.12), which has allowed us to present a simple and practical velocity correction to the usual slow-roll trajectory, as defined in Eq. (3.18). Adding this correction is trivial and, as shown in Fig. 2, immediately kills the absolute error on $\Delta N(\phi)$ by an order of magnitude, for all the tested models. We have also discussed additional improvements to better determine the field value at which inflation ends. This end-point correction does not necessarily perform better than the velocity correction alone as it breaks some fortuitous compensation of some neglected higher order terms. Nonetheless, it never degrades significantly the velocity-corrected trajectory and always allows for a more accurate determination of ρ_{end}.

Various other new results have been obtained along the course of searching for slow-roll improvements, such as the derivation of new integral constraints in Section 3.1, and, an exact, but still formal, new solution for the field trajectory when the functional $V(N)$ is known, see Appendix A. Our work could be improved in various directions, such as estimating the next terms in the expansion of Eq. (3.18), or, devising more involved methods to determine $\phi_{\rm end}$. However, one should keep in mind that more involved analytical methods are relevant only if they remain simpler, and numerically much faster, than bruteforcely integrating Eq. (2.7). The present work may be precisely filling this niche.

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A Formal solution

Starting from Eq. (2.7), instead of trying to use the field value ϕ as a variable, one may switch to the number of e-folds N and express the right-hand side as

$$\frac{\mathrm{d}\ln V}{\mathrm{d}\phi} = \frac{1}{\Gamma(N)}\frac{\mathrm{d}\ln V}{\mathrm{d}N}\,, \tag{A.1}$$

where the potential $V(\phi)$ is now viewed as a $V(N) = V[\phi(N)]$. Plugging Eq. (A.1) into Eq. (2.7), multiplying both sides by $\Gamma(6-\Gamma^2)$ one gets a differential equation for Γ^2

$$\frac{\mathrm{d}\Gamma^2}{\mathrm{d}N} + \left(6-\Gamma^2\right)\left(\Gamma^2 + \frac{\mathrm{d}\ln V}{\mathrm{d}N}\right) = 0. \tag{A.2}$$

This differential equation is a non-homogeneous Riccati equation and can be solved analytically [97]. Let us define the new "boost" function

$$\Lambda(N) \equiv \frac{1}{6-\Gamma^2(N)}\,. \tag{A.3}$$

In terms of $\Lambda(N)$, Eq. (A.2) considerably simplifies into

$$\frac{\mathrm{d}\Lambda}{\mathrm{d}N} + \left[6 + \frac{V'(N)}{V(N)}\right]\Lambda = 1, \tag{A.4}$$

This is a non-homogeneous linear differential equation which admits the exact solution

$$\Lambda(N) = e^{-6\Delta N}\frac{V_{\rm end}}{V(N)}\Lambda_{\rm end} - \int_N^{N_{\rm end}} e^{6(n-N)}\frac{V(n)}{V(N)}\,\mathrm{d}n\,, \tag{A.5}$$

where, as before, $\Delta N \equiv N - N_{\rm end}$. This solution extends an approximated one derived in Ref. [18] under the "non-relativistic" assumption ($\Gamma^2 \ll 6$).

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